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On representing observables in quantum mechanics

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Abstract

There are self-adjoint operators which determine both spectral and semispectral measures. These measures have very different commutativity and covariance properties. This fact opens additional possibilities with regard to the physical meaning of such a self-adjoint operator and its associated operator measures.

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1. Introduction

It is well known that a given self-adjoint operator may occur as the first moment operator of various semispectral measures, including its unique spectral measure. It is, perhaps, less widely known that there are self-adjoint operators which uniquely determine not only their spectral measures but also their semispectral measures. This situation requires us to extend the traditional text book wisdom of quantum mechanics whereby physical quantities, also called observables, are represented as self-adjoint operators (only).

In this paper, we wish to draw attention to the necessary extension by means of examples. To avoid an early commitment to a particular approach to quantum observables, in the main body of the paper we use the standard mathematical terminology of self-adjoint operators, spectral measures and semispectral measures. The discussion on the physical meaning of the mathematical formalism is postponed till the final section of the paper.

The use of semispectral measures (normalized positive operator-valued measures) both in analysing actual experiments and in studying conceptual and mathematical foundations of quantum mechanics has increased greatly during the last three decades, as can be seen by the appearance of a number of monographs on the subject, *qv* [1–8] amongst others.

Though not exclusively, the need to use semispectral instead of spectral measures (projection-valued measures) is often explained, explicitly or implicitly, as resulting from some uncontrollable aspects or statistical decisions. The results of this paper support the view that semispectral measures have a fundamental role in quantum mechanics beyond this. For one

thing, a semispectral measure can be assigned to observables whose ‘spectrum’ is, say, a curved surface (of moderate regularity) which is considerably more difficult to describe in purely operator theoretic terms.

2. Description of the problem

Let A be a self-adjoint operator, with a domain of definition $\mathcal{D}(A)$, and let $E : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ be a spectral measure defined on the Borel subsets of the real line \mathbb{R} and taking values on the set $\mathcal{L}(\mathcal{H})$ of bounded operators on \mathcal{H} .

For any two vectors $\varphi, \psi \in \mathcal{H}$, we let $E_{\varphi, \psi}$ denote the complex measure $X \mapsto E_{\varphi, \psi}(X) := \langle \varphi | E(X) \psi \rangle$. According to the spectral theorem for self-adjoint operators, any spectral measure E determines a unique self-adjoint operator A , with the domain $\mathcal{D}(A)$, such that for any $\varphi \in \mathcal{H}, \psi \in \mathcal{D}(A)$,

$$\langle \varphi | A \psi \rangle = \int_{\mathbb{R}} x \, dE_{\varphi, \psi}(x) \quad (1)$$

$$\mathcal{D}(A) = \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} x \, dE_{\varphi, \psi}(x) \text{ exists for all } \varphi \in \mathcal{H} \right\} \quad (2)$$

$$= \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} x^2 \, dE_{\psi, \psi}(x) < \infty \right\} \quad (3)$$

and, conversely, any self-adjoint operator A determines a unique spectral measure E such that the above relations are valid. We let E^A stand for the spectral measure of A , and we note that A is the first moment operator of the operator measure E^A .

Due to the multiplicativity of the spectral measure, the k th moment operator $E^A[k]$ of E^A is the k th power of its first moment operator A . That is, for any $k \in \mathbb{N}$,

$$E^A[k] := \int_{\mathbb{R}} x^k \, dE^A(x) = \left(\int_{\mathbb{R}} x \, dE^A(x) \right)^k = (E^A[1])^k = A^k$$

where the operator equalities are in the weak sense, as in (1), with a definition for $\mathcal{D}(A^k)$ analogous to that given in (3). It is also well known that the spectrum of A , $\sigma(A)$, is equal to the support, $\text{supp}(E^A)$, of E^A :

$$\sigma(A) = \text{supp}(E^A).$$

In two recent papers [9, 10] it was independently shown that there are self-adjoint operators A which both uniquely determine and are determined by certain semispectral measures $F : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ such that for all $\varphi \in \mathcal{H}, \psi \in \mathcal{D}(A)$,

$$\langle \varphi | A \psi \rangle = \int_{\mathbb{R}} x \, dF_{\varphi, \psi}(x) \quad (4)$$

$$\mathcal{D}(A) = \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} x \, dF_{\varphi, \psi}(x) \text{ exists for all } \varphi \in \mathcal{H} \right\} \quad (5)$$

$$\supseteq \left\{ \psi \in \mathcal{H} \mid \int_{\mathbb{R}} x^2 \, dF_{\psi, \psi}(x) < \infty \right\}. \quad (6)$$

In other words, A is the first moment operator of the operator measure F , and F is uniquely determined by A . Since F is not the spectral measure, the set inclusion (6) may, in general, be a proper one⁴.

The self-adjoint operators A in question are of a special type, specifying and being specified by semispectral measures F with particular additional properties. In spite of this mutual specification, however, the k th moment operator will not be the k th power of the first moment operator in general. In particular, it will be the case that

$$F[2] = \int x^2 dF(x) \geq \left(\int x dF(x) \right)^2 = F[1]^2 = A^2.$$

It must be stressed that it is quite exceptional for a semispectral measure F to be determined by its first moment operator $F[1] = A$. For a general semispectral measure F , even the knowledge of the moment operators $F[k]$ for all $k \in \mathbb{N}$ will not suffice to determine F .

However, if the support of F is compact then its moment operators $F[k], k \geq 0$, are bounded self-adjoint operators and the operator sequence $F[k], k \geq 0$, determines the operator measure F .⁵ Clearly, the spectrum of A is then a subset of the support of F ,

$$\sigma(A) \subseteq \text{supp}(F)$$

with the possibility that the inclusion is a proper one.

Given this background, we now state the extended viewpoint for quantum measurement theory that we referred to above.

According to the usual text book formulation of quantum mechanics, physical quantities are represented by self-adjoint operators, and, usually, even the converse is assumed (if no superselection rules are involved): each self-adjoint operator corresponds to a physical quantity. The mathematics just described raises the following question: if a given self-adjoint operator A gives rise to a unique spectral measure E^A and (by the formula prescribed in [9, 10]) a unique semispectral measure F , and these two measures are not the same, i.e., $E^A \neq F$, what is the relationship of the operator A and the two measures E^A and F , to the observable? If the observable is represented by the self-adjoint operator A , what is to be understood by the differing measure representations E^A and F , that it has? If, on the other hand, observables are completely represented by measures, then what is to be made of the fact that the self-adjoint operator A is now associated with two distinct observables, E^A and F ? We shall investigate these questions by considering three sets of examples.

The first two of them give examples of the situation described above, whereas the third illustrates the commonly accepted viewpoint that some semispectral measures associated with a self-adjoint operator are to be interpreted as smeared, noisy, unsharp or inaccurate versions of the observable represented by the spectral measure of the self-adjoint operator.

3. Examples

The examples that illustrate the problem arise from the theory of generalized imprimitivity systems, also called systems of covariance. The primary examples were discussed in [9, 10]. Here we follow [12–14] to provide somewhat wider classes of relevant examples.

⁴ We recall that it is the positivity of the operator measure F , that is, $F(X) \geq 0$ for all $X \in \mathcal{B}(\mathbb{R})$, which implies that the square integrability domain of equation (6) is a subspace of $\mathcal{D}(A)$. If, in addition, F were projection valued, that is, $F(X) = F(X)^2$ for all $X \in \mathcal{B}(\mathbb{R})$, then the set inclusion (6) would be an equality [11, lemma A.2], cf equation (3).

⁵ This is a well-known consequence of the Weierstrass approximation theorem and the uniqueness part of the Riesz representation theorem.

3.1. \mathbb{N} -covariant semispectral measures

Let \mathcal{H} be a complex separable Hilbert space, $\{|n\rangle\}_{n \in \mathbb{N}}$ a fixed orthonormal basis for \mathcal{H} , and let N denote the self-adjoint operator for which $N|n\rangle = n|n\rangle$ for all $n \in \mathbb{N}$.

Consider a semispectral measure F defined on the Borel subsets of the interval $[0, 2\pi)$ and taking values in $\mathcal{L}(\mathcal{H})$. We say that F is \mathbb{N} -covariant if it forms together with the unitary representation $x \mapsto e^{ixN}$, $x \in \mathbb{R}$, of the additive group of \mathbb{R} , a generalized imprimitivity system, that is

$$e^{ixN} F(X) e^{-ixN} = F(X + x) \quad (7)$$

for all $X \in \mathcal{B}([0, 2\pi))$, $x \in \mathbb{R}$, where the addition $X + x$ is modulo 2π . (The labelling of the covariance by \mathbb{N} refers to the spectrum of N .)

The structure of the \mathbb{N} -covariant semispectral measures $F : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$ is well known (see, e.g., [2, 15]). Perhaps, the simplest way to characterize them is the following [12]: F satisfies the covariance condition (7) if and only if there is a (not necessarily orthogonal) sequence of unit vectors $(h_n)_{n \in \mathbb{N}}$ of \mathcal{H} such that for any $X \in \mathcal{B}([0, 2\pi))$,

$$F(X) = \sum_{n, m \in \mathbb{N}} \langle h_n | h_m \rangle \frac{1}{2\pi} \int_X e^{i(n-m)x} dx |n\rangle \langle m| \quad (8)$$

where the series converges weakly.

It is to be noted that two sequences of unit vectors $(h_n)_{n \in \mathbb{N}}$ and $(h'_n)_{n \in \mathbb{N}}$ determine the same semispectral measure F exactly when $\langle h_n | h_m \rangle = \langle h'_n | h'_m \rangle$ for all $n, m \in \mathbb{N}$.

By a direct computation, one may easily confirm that the only commutative solution of (7) is the scalar measure: the commutativity of F , that is, $F(X)F(Y) = F(Y)F(X)$ for all $X, Y \in \mathcal{B}([0, 2\pi))$, holds if and only if the generating vectors h_n are pairwise orthogonal; in that case $F(X) = \frac{1}{2\pi} \int_X dx I$ for all $X \in \mathcal{B}([0, 2\pi))$.⁶ In particular, this means that among the \mathbb{N} -covariant semispectral measures (8) there is no spectral measure.

The \mathbb{N} -covariant semispectral measures $F : \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$ are supported by the interval $[0, 2\pi]$. Therefore, their moment operators,

$$\begin{aligned} F[k] &= \int_0^{2\pi} x^k dF(x) \quad k \in \mathbb{N} \\ &= \sum_{n, m \in \mathbb{N}} \langle h_n | h_m \rangle \frac{1}{2\pi} \int_0^{2\pi} x^k e^{i(n-m)x} dx |n\rangle \langle m| \end{aligned} \quad (9)$$

are bounded self-adjoint operators. Since no such F is projection valued, as noted above, the second moment operator $F[2]$ is never equal to the square of the first moment operator $F[1]$,

$$F[2] \geq F[1]^2 \quad F[2] \neq F[1]^2 \quad (10)$$

(see, for instance, [17, appendix, equation (9), p 446]).

The covariance condition (7) completely determines the structure of the semispectral measures (8) and thus also their moment operators (9). But from (9) we see that

$$\langle h_n | h_m \rangle = \langle n | F[1] | m \rangle i(n - m) \quad n \neq m \quad (11)$$

which implies that any \mathbb{N} -covariant semispectral measure F is uniquely determined by its first moment operator $F[1]$,

$$F[1] = \pi I + \sum_{n \neq m \in \mathbb{N}} \frac{\langle h_n | h_m \rangle}{i(n - m)} |n\rangle \langle m| \quad (12)$$

(see [9, 10]).

⁶ For a full analysis of the degree of commutativity of the \mathbb{N} -covariant semispectral measures F (see [16]).

As a bounded self-adjoint operator, $F[1]$ has a unique spectral measure $E^{F[1]}$ such that

$$F[1] = \int_{\mathbb{R}} x \, dE^{F[1]}(x) \tag{13}$$

where the integral ranges effectively over the spectrum of $F[1]$. The operator $F[1]$ thus determines both a unique \mathbb{N} -covariant semispectral measure F and a unique spectral measure $E^{F[1]}$. A distinctive feature is that the spectral measure $E^{F[1]}$ cannot be \mathbb{N} -covariant. Also, apart from the trivial case, F is noncommutative whereas $E^{F[1]}$ is multiplicative and thus commutative. We further note that $\text{supp}(E^{F[1]}) = \sigma(F[1]) \subseteq \text{supp}(F)$.

Perhaps the most natural and important example of the \mathbb{N} -covariant semispectral measures F is the one associated with a constant sequence $h_n = h, n \in \mathbb{N}$. (Any unit vector will do; see above.) This semispectral measure has been advocated by some authors as the canonical phase observable F_{can} (see, e.g., [2, 6, 9]). However, since its first moment is not canonically conjugate to the number operator, employing the word ‘canonical’ in this context is not the familiar textbook usage; canonicity here is with respect to the above class of semispectral measures.

Similarly, its first moment operator⁷

$$F_{\text{can}}[1] = \pi I + \sum_{n \neq m=0}^{\infty} \frac{1}{m-n} |n\rangle\langle m| \tag{14}$$

is frequently proposed as the phase operator (see, e.g., [18–20]). In this case, $\sigma(F_{\text{can}}[1]) = \text{supp}(F_{\text{can}})$. The spectral measure of $F_{\text{can}}[1]$ has a rather complicated structure (see [18] for an analysis); nevertheless it is not \mathbb{N} -covariant. This and other candidate phase observables are discussed at length in [24].

3.2. \mathbb{Z} -covariant semispectral measures

Taking an orthonormal basis labelled by the set of all, and not simply non-negative, integers leads to a class of examples similar to those obtained in section 3.1. Some new and interesting features do arise with this choice, however. Therefore, let $\{|k\rangle\}_{k \in \mathbb{Z}}$ be an orthonormal basis of \mathcal{H} and let Z denote the self-adjoint operator with $Z|k\rangle = k|k\rangle$ for all $k \in \mathbb{Z}$.

Extending the previous terminology, we say that a semispectral measure $F: \mathcal{B}([0, 2\pi)) \rightarrow \mathcal{L}(\mathcal{H})$ is \mathbb{Z} -covariant if it satisfies the covariance condition

$$e^{ixZ} F(X) e^{-ixZ} = F(X + x) \tag{15}$$

for all $X \in \mathcal{B}([0, 2\pi))$, $x \in \mathbb{R}$, where the addition $X + x$ is modulo 2π .

As in section 3.1, a semispectral measure F is \mathbb{Z} -covariant if and only if there is a sequence of unit vectors $(h_k)_{k \in \mathbb{Z}}$ of \mathcal{H} such that for any $X \in \mathcal{B}([0, 2\pi))$

$$F(X) = \sum_{k,l \in \mathbb{Z}} \langle h_k | h_l \rangle \frac{1}{2\pi} \int_X e^{i(k-l)x} \, dx |k\rangle\langle l|. \tag{16}$$

The principal difference here is that among the solutions (16) of the covariance condition (15) there are both commutative and noncommutative semispectral measures and, in addition, a spectral measure (unique up to unitary equivalence) obtained with an arbitrary choice of unit vector $h_k = h$ for all $k \in \mathbb{Z}$. For each solution F , the first moment operator $F[1]$ uniquely determines the semispectral measure F .⁸

⁷ This operator is unitarily equivalent to the Toeplitz operator of multiplication by the independent variable on the Hardy–Hilbert subspace of square integrable functions on the circle.

⁸ The structure of the moment operators are like in the \mathbb{N} -covariant case, with the sole exception that now the summations are over \mathbb{Z} (cf [12]).

Clearly, the spectral measure $E^{F[1]}$ of $F[1]$ is \mathbb{Z} -covariant exactly when it coincides with F , which is now the case for the constant sequence $h_k = h, k \in \mathbb{Z}$. In this case, the pair $(F_{\text{can}}[1], Z)$ constitutes a Schrödinger pair, that is, the usual position–momentum operators of a particle in a box of length 2π .

3.3. \mathbb{R} -covariant semispectral measures

To emphasize the very special nature of the previous two sets of examples, let us consider next the multiplicative operator Q in $L^2(\mathbb{R})$, $(Q\varphi)(x) = x\varphi(x)$, with the domain $\mathcal{D}(Q) = \{\varphi \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^2 |\varphi(x)|^2 dx < \infty\}$. Consider the unitary representation $x \mapsto U_x$ of the real line \mathbb{R} given by $(U_x\varphi)(y) = \varphi(y - x)$. As is well known, the spectral measure of Q , E^Q , is (up to unitary equivalence) the unique projection-valued solution of the \mathbb{R} -covariance condition [21]

$$U_x F(X) U_x^* = F(X + x) \quad X \in \mathcal{B}(\mathbb{R}) \quad x \in \mathbb{R}. \quad (17)$$

However, this covariance condition (17) can be solved for arbitrary semispectral measures [2], and one obtains thereby both commutative and noncommutative semispectral measures in addition to the spectral measure [14]. In particular, any convolution of E^Q with a probability density f yields an \mathbb{R} -covariant semispectral measure of the form $E^{Q,f}(X) = (\chi_X * f)(Q)$, $X \in \mathcal{B}(\mathbb{R})$, $f = |\eta|^2$, $\eta \in L^2(\mathbb{R})$, $\|\eta\| = 1$. If the expectation value of the density function f is zero, then the first moment operator of $E^{Q,f}$ equals the first moment operator of E^Q , namely Q (see, e.g., [1]). Therefore, Q cannot determine $E^{Q,f}$, so the dichotomy possible for \mathbb{T} -covariant systems cannot occur for this class of \mathbb{R} -covariant measures.

4. Discussion

In the approach to quantum mechanics which starts with the operational idea of a preparation and registration procedure, one is led in a natural way to the set of states and the set of observables being in duality. The states may be defined as equivalence classes of preparations, the observables as totalities of measurement outcome statistics.

States may be represented as positive normalized trace class operators ρ (known as density operators) on the configuration Hilbert space \mathcal{H} for the system under consideration. An observable may then be defined as a normalized semispectral measure F defined on the relevant (σ -algebra of subsets of) space of values (measurement outcomes), typically the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ of the real line \mathbb{R} (see, e.g., [1–3, 6]). The probability measure $X \mapsto F_\rho(X) := \text{Tr}[\rho F(X)]$, defined by a state ρ and an observable F , is then taken to describe the measurement outcome statistics obtained when the same F -measurement is repeated under the same conditions, described by ρ , a large number of times.

In this approach, spectral measures appear as special idealized cases, called decision observables in [3], ordinary observables in [4] and sharp observables in [6]. The first moment operator $F[1]$ of an observable F accounts for the expectation value $\text{Tr}[\rho F[1]]$ of the probability distribution F_ρ . The knowledge of the expectation values $\text{Tr}[\rho F[1]]$, for all states ρ , determines the operator $F[1]$ and thus its spectral measure $E^{F[1]}$ as well. In general, it does not determine the semispectral measure F . But the examples of sections 3.1 and 3.2 show that there are cases where this does happen.

To help analyse the question of the physical meaning of the self-adjoint operator $F[1]$ and its spectral measure $E^{F[1]}$ for \mathbb{N} -/ \mathbb{Z} -covariant semispectral measures F , we consider the resulting probability measures and their variances.

From the probabilistic point of view, the spectral and semispectral measures associated with $F[1]$ are quite different. For while the probability distributions $F_{\varphi,\varphi}$ and $E_{\varphi,\varphi}^{F[1]}$ in any vector state $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, have the same expectations

$$\int_{\sigma(F[1])} x \, dE_{\varphi,\varphi}^{F[1]}(x) = \langle \varphi | F[1] \varphi \rangle = \int_0^{2\pi} x \, dF_{\varphi,\varphi}(x) \tag{18}$$

their other moments are different. In particular, their variances are different:

$$\begin{aligned} \text{Var}(F_{\varphi,\varphi}) &= \int x^2 \, dF_{\varphi,\varphi}(x) - \left(\int x \, dF_{\varphi,\varphi}(x) \right)^2 \\ &= \langle \varphi | F[2] \varphi \rangle - \langle \varphi | F[1] \varphi \rangle^2 \\ &= \langle \varphi | F[2] \varphi \rangle - \langle \varphi | F[1]^2 \varphi \rangle + \langle \varphi | F[1]^2 \varphi \rangle - \langle \varphi | F[1] \varphi \rangle^2 \\ &= \langle \varphi | (F[2] - F[1]^2) \varphi \rangle + \text{Var}(E_{\varphi,\varphi}^{F[1]}) \\ &\geq \text{Var}(E_{\varphi,\varphi}^{F[1]}). \end{aligned} \tag{19}$$

This relation is sometimes taken to suggest that F could be regarded as a smeared or noisy version of $E^{F[1]}$, since, in a vector state φ , the variance of F is greater by a noise term $\langle \varphi | (F[2] - F[1]^2) \varphi \rangle$ than the variance of $E^{F[1]}$.

A smearing of, or noisy version of, $E^{F[1]}$ is typically obtained by convolving it with a density function f , that is, a nonnegative Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$, possibly supported by $[0, 2\pi]$, such that $\int f(x) \, dx = 1$ (see, e.g., [22, 6]). The semispectral measure $E^{F[1],f} : X \mapsto (\chi_X * f)(F[1])$ obtained in this way is commutative⁹.

All the (nontrivial) \mathbb{N} -covariant semispectral measures of section 3.1 are noncommutative. In contrast, convolutions of spectral measures with probability densities are commutative semispectral measures and none of them is \mathbb{N} -covariant. Note also that if the average of a smearing function f is zero, $\int x f(x) \, dx = 0$, then the first moment operator of $E^{F[1],f}$ is again $F[1]$, but, clearly, it cannot determine $E^{F[1],f}$.

We conclude that the self-adjoint operators $F[1]$ of section 3.1 constitute examples of self-adjoint operators which represent different observables F and $E^{F[1]}$. Their measurement outcome statistics, described by the probability measures $F_{\varphi,\varphi}$ and $E_{\varphi,\varphi}^{F[1]}$, are different, though they are indistinguishable by the statistical average. Their difference becomes evident, for instance, in their standard deviations. From the statistical point of view one may say that $E^{F[1]}$ is the observable associated with $F[1]$ which has the least variance [17], whereas F is the observable associated with $F[1]$ which is \mathbb{N} -covariant.

Among the solutions of \mathbb{Z} -covariant semispectral measures there are also commutative measures, including the canonical spectral measure F_{can} . However, convolutions of F_{can} which have $F_{\text{can}}[1]$ as the first moment cannot be \mathbb{Z} -covariant. Therefore, also in this case, the (noncovariant) spectral measure and the (covariant) semispectral measures of the self-adjoint moment operators $F[1]$ seem to represent different, though in the sense of the statistical mean, indistinguishable observables. It is also worth noting that among the commutative \mathbb{Z} -covariant semispectral measures F there is no smearing of F_{can} which would have the same first moment as F_{can} .

Finally, section 3.3 gives examples of \mathbb{R} -covariant semispectral measures which can be interpreted as smeared or unsharp versions of the sharp \mathbb{R} -covariant spectral measure E^Q . In that case, however, the first moment operator Q of an \mathbb{R} -covariant semispectral measure $E^{Q,f}$ does not suffice to determine the whole semispectral measure. We recall from [1, theorem 3.3.2] that if the density function f has finite mean and variance, then

⁹ In fact, any commutative semispectral measures can be represented as a probability average of a unique spectral measure [23] (see also [8, section 2.1.3.] and [7, theorem 3.1.3]).

$\text{Var}(E_{\varphi,\varphi}^{Q,f}) = \text{Var}(E_{\varphi,\varphi}^Q) + \text{Var}(f)$ for any sufficiently smooth vector states $\varphi \in L^2(\mathbb{R})$, that is, the noise term $\langle \varphi | E_{\varphi,\varphi}^{Q,f}[2] - E_{\varphi,\varphi}^{Q,f}[1]^2 | \varphi \rangle$ is then simply $\text{Var}(f)$.

Apart from the results detailed above, some examples of this kind go against too facile an understanding of how symmetries of a quantum-mechanical system show up. It would seem to be the established wisdom amongst physicists that if a quantum-mechanical system has a certain symmetry (in the usual sense), then this must be reflected in the experimental arrangements for the system. Conversely, if recorded data evidences a symmetry, that symmetry must be present in the system. The opposite is also felt to hold: if the system has no symmetry, no symmetry should show up in the data and conversely.

Reasonable as this seems, when it comes to turning these statements into mathematical terms, our examples show that things are not as cut and dried as this. To see that, consider once more a simple case of an \mathbb{N} -covariant semispectral measure and its first moment operator taken from [10]. Fix a pair of non-negative integers s, t and a complex number $z = |z| e^{i\varphi}$, with $|z| \leq 1$. The semi-spectral measure in question is, for every Borel subset $X \subseteq [0, 2\pi)$,

$$F(X) = \frac{1}{2\pi} \int_X dx + \frac{z}{2\pi} \int_X e^{i(t-s)x} dx |s\rangle\langle t| + \frac{\bar{z}}{2\pi} \int_X e^{-i(t-s)x} dx |t\rangle\langle s|. \quad (20)$$

The spectral decomposition of the first moment operator of F is now simply

$$F[1] = \pi P_\pi + \left(\pi + \frac{|z|}{t-s} \right) P_+ + \left(\pi - \frac{|z|}{t-s} \right) P_- \quad (21)$$

where

$$\begin{aligned} P_+ &= \frac{1}{2} [|s\rangle\langle s| + |t\rangle\langle t| + i(e^{-i\varphi} |t\rangle\langle s| - e^{i\varphi} |s\rangle\langle t|)] \\ P_- &= \frac{1}{2} [|s\rangle\langle s| + |t\rangle\langle t| - i(e^{-i\varphi} |t\rangle\langle s| - e^{i\varphi} |s\rangle\langle t|)] \\ P_\pi &= I - |s\rangle\langle s| - |t\rangle\langle t|. \end{aligned}$$

Thus, $F[1]$ is seen to have three eigenvalues, of which π is infinitely degenerate and the other two non-degenerate.

The probability measures for the states of the system obtained from F show a continuous symmetry, translations in $[0, 2\pi)$ (mod 2π), but this symmetry is not possessed by the first moment operator $F[1]$. Moreover, this is not simply a matter of F being an inferior measurement scheme for $F[1]$, since one can use the data obtained from these F -probability measures to (mathematically) reconstruct $F[1]$. Conversely, were one to measure $F[1]$ and obtain the complete spectral information, one could (mathematically) construct F . In terms of quantum-mechanical information obtainable in principle, therefore, they are equivalent and complete (in the Copenhagen sense). We, at any rate, find this surprising.

Sufficiently so that one must wonder if this is a phenomenon associated only with the systems we have considered. Is there something like this associated with maximal, but not self-adjoint, operators such as the ‘momentum’ operator on $L^2([0, \infty))$? Also the case of \mathbb{R} -covariant systems calls for further studies in that respect. There, work is in progress but a report will have to wait for another publication in the near future.

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